Supplementary Materials for "The statistics of frictional families"

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In the Supplementary Materials, we provide additional details concerning two technical points made in the manuscript. We first describe results from numerical simulations of packings of three equal-sized frictionless disks to illustrate the difference between 'basin volumes' (*i.e.* the collection of initial conditions in configuration space that map to each distinct packing [1]) and the total volume in configuration space occupied by a given class of the saddle packings with m missing contacts. We also include a discussion of how rattler particles affect the form of the coefficients in the power series expansion of the partition function in terms of the static friction coefficient.

1. Probability for a specific packing versus the fraction of mth order saddle packings We focus on a system with three equal-sized frictionless disks confined within a 1x1 square box with fixed walls. For this system, there is only one mechanically stable packing with $m = 0$ up to particle rotations, inversions, and label exchanges. Since the system has fixed instead of periodic boundary conditions, the isostatic number of contacts is $N_c^0 = 2N+1 = 7$. If we treat packings that are related by rotations, inversions, and label exchanges separately, there are 24 microstate packings with $m = 0$. Using the successive compression and energy relaxation packing generation protocol starting at zero packing fraction [2], we measured the probability for obtaining each of the $m = 0$ packings with the constraint that particles 2 and 3 start at positions (0.2, 0.6) and (0.45, 0.85), respectively, and particle 1 is initialized at location (x,y) in the box.

The position of each pixel in the box in Fig. 1 represents the initial position of particle 1 and its color corresponds to one of the 11 out of 24 microstates to which the system evolved using the packing generation protocol. (The remaining 13 microstates can not be obtained using this particular set of initial conditions.) The white lines indicate the location of particle 1 in its final position for each microstate. The area of each colored region gives the probability of finding that particular microstate. This area is unrelated to the area in configuration space of each final $m = 0$ packing, which is a small circular area whose diameter decreases with increasing accuracy of identifying the location of the packing. We see explicitly that the Gibbs equal-probability assumption for particular $m = 0$ microstates does not hold since the basin areas of the microstates are not equal.

In Fig. 1, we also identity (by a solid black line) saddle packings with $m = 1$ that connect each of the 11 final microstate $m = 0$ packings. One example of an $m = 1$ saddle packing (shaded gray) is shown in the bottom right of the box and the position of particle 1 in this microstate is indicated. In the closely related microstate $m = 0$ packing (colored orange), particle 1 is touching the bottom boundary. In the microstate $m = 1$ packing, particle 1 has moved away from the wall. Near the top right corner of the collection of $m = 1$ packings, particle 2 touches the bottom wall creating another microstate $m = 0$ packing (colored red). This process continues as the collection of $m = 1$ packings is traversed in configuration space (as one moves along the black solid line) with different particles leaving and then touching the fixed walls. To determine the probability of having a final microstate packing with $m = 0$ or 1 missing contacts, the shaded regions are irrelevant. We know that the system must occur in configuration space on the black line, and we can use the Gibbs' assumption to determine the fraction of microstate packings that will occur within a characteristic distance δ of the $m = 0$ points or $m = 1$ lines. (See Eq. 1 in the main text.)

2. Coefficients in the power series expansion of the partition function in terms of the static friction coefficient For N frictionless particles in systems with periodic boundary conditions, $m = 0$ packings have $N_c = N_c^0 = 2(N - N_r) - 1$ contacts in the force-bearing backbone, where N_r is the number of rattler particles. $m = 1$ packings possess $N_c^0 - 1$ contacts and mth order saddle packings possess $N_c^0 - m$ contacts. As a zeroth order model, we assume 1) N_r is constant for all higher-order packings that originate from a given $m = 0$ packing and 2) any one of the $N_c^0 - m$ contacts can be removed for each m up to $m_{\max}$.

With these assumptions, $a_m \propto N_b(N, m)$ (the number of branches at saddle order m) can be calculated from the fact that for every $m = 0$ packing there are N_c^0 ways to make an $m = 1$ packing with $N_c^0 - 1$ contacts. For every $m = 1$ packing, there are $N_c^0 - 1$ ways to make an $m = 2$ packing with $N_c^0 - 2$ contacts. This pattern repeats until there are not enough contacts to stabilize the packing even in the limit $\mu \to \infty$. Thus, with these assumptions, there are $N_b(N, m) = N_c^0(N_c^0 - 1) \dots (N_c^0 - m) = C_m^{N_c^0}$ packings that stem from each $m = 0$ packing. This form for a_m captures all of the qualitative features of the probability $P_m(\mu)$ to have m missing contacts at static friction coefficient μ. However, it does not match the results

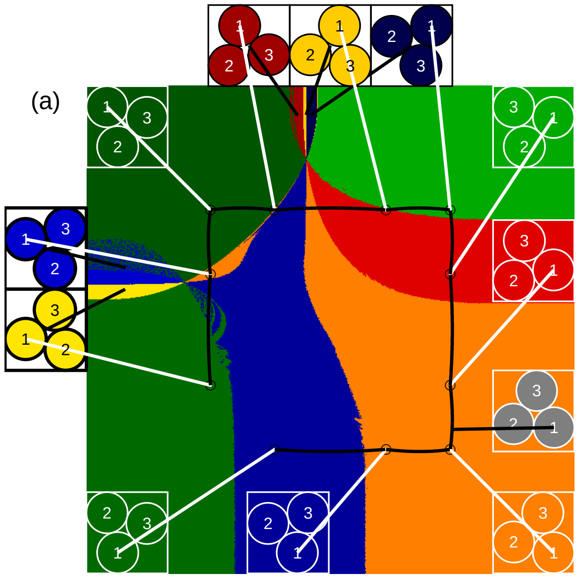

FIG. 1: Illustration of the basin areas for 11 of the microstate $m = 0$ packings for a system with three equal-sized frictionless disks confined within a 1x1 box with the positions of particles 2 and 3 initially set to (0.2,0.6) and (0.45,0.85), respectively. The color of each pixel at (x,y) within the box indicates the final microstate $m = 0$ packing to which the system evolved when particle 1 is initially placed at location (x,y). The locations of particle 1 in the final packings are indicated by the black solid line. Microstate $m = 1$ packings link sucessive $m = 0$ packings along the blacik solid line. An example microstate $m = 1$ packing is highlighted in gray.

from the MD simulations of the Cundall-Strack model quantitatively. (See Fig. 3 (a) in the main text).

Let us examine in detail the two assumptions in the zeroth order model. First, the number of rattlers clearly can change as contacts are removed. For example, in an $m = 0$ packing every non-rattler particle has at least 3 contacts (and on average 4 contacts), and so removing one contact will not create any new rattler particles. However, when the next contact is removed, particles that had 3 contacts in the $m = 0$ packing will become rattlers along the branch in which we remove 2 contacts from the same particle. On an $m = 2$ branch, where a particle with 3 contacts loses 2 of the contacts, the third contact is not considered a true contact. In this case, the number of contacts is $N_c^{'} = N_c - 3 = 2(N - N_r) - 1 - 3 = 2(N - N_r) - 4$ and, because there are now $N_r^{'} = N_r + 1$ rattler particles, $(N_c^0)^{'} = 2(N - (N_r + 1)) - 1 = 2(N - N_r) - 3$ and $m = (N_c^0)^{'} - N_c^{'} = 1$. Therefore, N_3 out of the $N_c^0(N_c^0 - 1)(N_c^0 - 2)$ $m = 2$ branches for an $m = 0$ packing with N_3 particles with three contacts will be transformed into $m = 1$ packings. These transformations of $m = 2$ packings into $m = 1$ packings create more $m = 1$ packings relative to $m = 2$ packings than given by the zeroth order approximation, $a_m = C_m^{N_c^0}$.

Packings with $m = 3$ can also be converted into $m = 2$ packings by creating rattler particles from the backbone. However, this trend cannot continue for all m. Whenever a contact is removed and a rattler particle is created, none of its contacts can remain part of contact network, so all contacts are removed. However, removing all of these contacts may create a new rattler particle, which in turn will create a new one, and cause the packing to become unstable. Creating a cascade where rattler particle are successively created becomes more likely as m approaches $m_{\max}$. Thus, we expect that for large m, the increase in the number of branches due to the creation of rattler particles will no longer dominate. We anticipate that $c_m(N) = a_m / C_m^{N_c^0}$ will increase from $m = 0$ to a maximum near $m_{\max}/2$ and then decrease again as m approaches $m_{\max}$.

Another geometrical feature that can affect the accuracy of the zeroth order model is that mth order packings can occur for which there is no progenitor $m = 0$ packing. These packings are not included in our current theoretical analyses because they are not created by removing contacts from $m = 0$ packings. A possible source for these packings could be that they are created by the inclusion of a rattler particle in an $m = 0$ packing. For example, a rattler particle could be brought into contact with two particles A and B. This would add 2 contacts and remove 1 rattler particle leaving an $m = 0$ packing that can only be stabilized by friction. If a contact was then removed between A and B, an $m = 1$ packing would be created with no connection to a frictionless $m = 0$ packing. As m increases, there are more ways to create such 'isolated' packings. However, as m nears $m_{\max}$ there will be fewer ways to create isolated packings since all of the remaining contacts are needed for the force-bearing contact network. Thus, we again expect $c_m(N)$ to be largests near $m_{\max}/2$.

These effects lead to the conclusion that we should expect that the coefficients of the power series expansion of the partition function relative to the zeroth order approximation, $c_m(N)$, will increase from $m = 0$ to a maximum near $m_{\max}/2$ and then decrease as m approaches $m_{\max}$. In Fig. 2, we plot a least-squares fit of $c_m(N) = a_m / C_m^{N_c^0}$ to the data from the numerical simulations of the Cundall-Strack model. We find that the deviation is approximately Gaussian in m with $c_m(N) \approx \exp(-m(m - m_{\max})/m_{\max})$. This finding is in agreement with our expectations of the likely deviations from the zeroth order model. The relative simplicity of this correction suggests that a more thorough accounting of the probability for saddle number transformation by rattler particles joining and exiting the force-bearing contact network will be possible, and we are currently working on these corrections, which will be presented in a forthcoming manuscript.

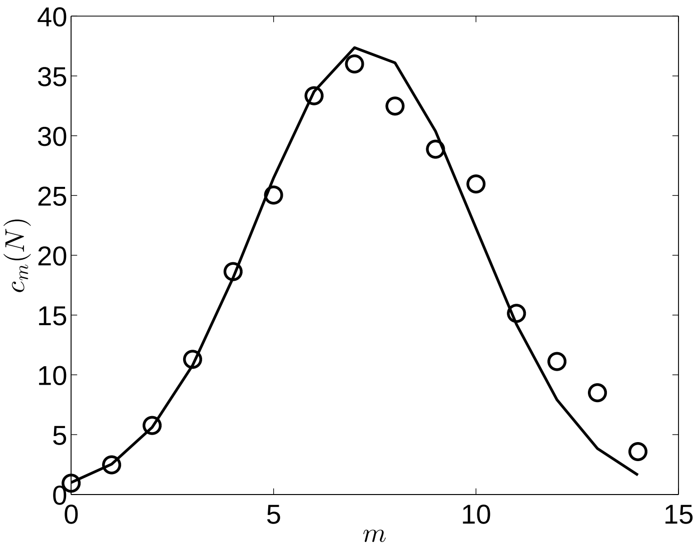

FIG. 2: The coefficients $c_m(N) = a_m(N)/C_m^{N_c^0}$ of the power series expansion of the partition function normalized by the values from the zeroth order model in terms of static friction coefficient obtained from least-squares fitting to the contact number data from the numerical simulations of the Cundall-Strack model (circles). The form $c_m(N) = \exp[-m(m - m_{\max})/m_{\max}]$ in Eq. 4 in the main text is indicated by the solid line.

[1] S. S. Ashwin, J. Blawzdziewicz, C. S. O'Hern, and M. D. Shattuck, *Phys. Rev. E* **85** (2012) 061307.
[2] G.-J. Gao, J. Blawzdziewicz, and C. S. O'Hern, *Phys. Rev. E* **74** (2006) 061304.