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Physics 35100 Mechanics
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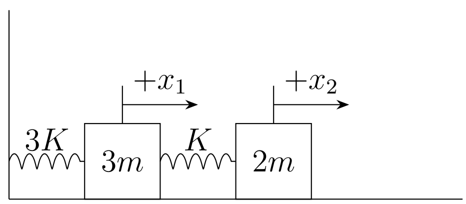
Problem Set 3

Question 1. *Double Oscillator* Two blocks of mass $m_1 = 3m$ and $m_2 = 2m$ are on a friction-less table. m_1 is connected to a linear spring with spring constant $K_1 = 3K$ and m_2 is connected to a second linear spring with spring constant $K_2 = K$. Both masses are confined to move in one dimension. The masses are initial at rest at their equilibrium positions. x_1 and x_2 measure their displacement from the equilibrium positions.



(1) What is the Kinetic Energy of the system in terms of x_1 and x_2.
(2) What is the Potential Energy of the system in terms of x_1 and x_2.
(3) Show that the potential energy of the system is $2K$ when $x_1 = 1$ and $x_2 = 0$. In this configuration, describe the state of the $3K$ and K springs as stretched, compressed, or at equilibrium.
(4) Find the equations of motion for the system.
(5) Express the equations of motion for the system as a matrix equation of the form $\mathsf{M}\ddot{\mathsf{X}} = -\mathsf{K}\mathsf{X}$, where M and K are 2×2 matrices and X is a 2×1 matrix.
(6) Show that $\mathsf{X}(t) = \mathsf{A}\cos(\omega t + \phi)$, where A is a 2×1 matrix, is a solution to the equation $\mathsf{M}\ddot{\mathsf{X}} = -\mathsf{K}\mathsf{X}$. What are the conditions on A, ω, and ϕ so that $\mathsf{X}(t) = \mathsf{A}\cos(\omega t + \phi)$ is a solution?
(7) Find the values of ω which satisfy $\det(\mathsf{K} - \omega^2\mathsf{M}) = 0$ or $\det(\mathsf{M}^{-1}\mathsf{K} - \omega^2\mathbf{I}) = 0$ for the K and M found above, and $\mathbf{I}$ is the identity matrix.
(8) For each value of ω find an A which satisfies $(\mathsf{K} - \omega^2\mathsf{M})\mathsf{A} = 0$, or $\mathsf{K}\mathsf{A} = \omega^2\mathsf{M}\mathsf{A}$, or $\mathsf{M}^{-1}\mathsf{K}\mathsf{A} = \omega^2\mathsf{A}$.
(9) Using the A's and ω's from above:
  (a) Find the general solution for $\mathsf{X}(t)$.
  (b) Show that in matrix form it can be expressed as:

$$\mathsf{X}(t) = \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C_1 \cos\left(\sqrt{\frac{3}{2}}\omega_0 t + \phi_1\right) \\ C_2 \cos\left(\sqrt{\frac{1}{3}}\omega_0 t + \phi_2\right) \end{bmatrix}$$

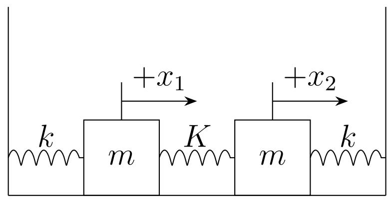
  where C_1 and C_2 are constants.
  (c) Show that it is a solution to the equations of motion.
(10) Show that the change of variables:

$$\mathsf{Y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix}^{-1} \mathsf{X}(t)$$

decouples the solution so that $y_1(t)$ and $y_2(t)$ oscillate independently, each with their own frequency and phase. Describe the motion when $C_1 = 1$ and $C_2 = 0$ and when $C_1 = 0$ and $C_2 = 1$.

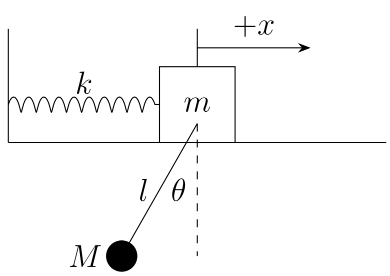
(11) Find C_1 and C_2 for the initial condition of $\mathsf{X} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\dot{\mathsf{X}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Explain in words what this initial condition represents.

Question 2. *Weak vs. Strong coupling* Two blocks of mass m are connected by springs as shown. The middle spring has a spring constant K and the springs connected to the walls have the same spring constant k.



(1) What are the normal modes and normal frequencies for this system?
(2) Describe the normal modes for three cases:
  (a) $K >> k$.
  (b) $K = k$.
  (c) $K << k$.

Question 3. *Oscillating pendulum* A simple pendulum of mass M and length l is hanging from a block with mass m that can oscillate at the end of a spring with spring constant k connected to a wall as shown.



(1) Find the Lagrangian $L(x, \theta, \dot{x}, \dot{\theta})$ under the assumption that the angle θ is small, so that $\sin\theta \simeq \theta$ and $\cos\theta \simeq 1 - \theta^2/2$. and only retain quadratic terms in $x, \theta, \dot{x}, \dot{\theta}$. (e.g., $x\theta$ is ok, but $x\theta\dot{\theta}$ is too small. and can be ignored.)
(2) Find the normal modes and corresponding frequencies for the case $m = M = l = g = 1$ and $k = k$.
(3) Describe the normal modes for three cases:
  (a) $k >> 1$.
  (b) $k \sim 1$.
  (c) $k << 1$.